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AN ORTHOTROPIC THEORY OF VISCOPLASTICITY BASED ON OVERSTRESS FOR THERMOMECHANICAL DEFORMATIONS

K.-D. LEE† and E. KREMPL
Mechanics of Materials Laboratory, Department of Mechanical Engineering, Aeronautical Engineering and Mechanics, Rensselaer Polytechnic Institute, Troy, NY 12180-3590, U.S.A.

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Abstract—An infinitesimal, orthotropic theory of viscoplasticity based on overstress for thermomechanical loading (TVBO) is presented. The total strain rate is the sum of elastic, inelastic and thermal strain rates. An orthotropic constitutive law is postulated for each strain rate using the characteristics of orthotropic matrices and previous isotropic formulations of the viscoplasticity theory as a guide. All material functions and constants can be functions of current temperature and no influence of temperature history is modeled. Yield surfaces and loading/unloading conditions are not used in the theory in which the inelastic strain rate is solely a function of the overstress, the difference between stress and the equilibrium stress, a state variable of the theory. A comparatively simple theory is obtained which is capable of modeling important phenomena like creep, relaxation, rate sensitivity, hysteresis, tension/compression asymmetry and nearly elastic regions. It is also possible to model quasielastic behavior in one direction while the others behave viscoplastically. The theory is shown to reduce to a previously proposed formulation for inelastic incompressibility and isotropy.

INTRODUCTION

Continuing demands to increase efficiency and economy of engineering structures require that the inelastic load carrying capabilities of materials be utilized. This is especially so for components of propulsion and power generation machinery which operate at elevated temperature. They have to be designed against creep and creep/fatigue failure due to the existence of nonlinear creep deformation and due to large and small temperature cycles which are repeated during the lifetime of a component. A prerequisite for a successful design is the ability to compute the time-dependent deformations as a function of location and time in severely loaded components.

For nominally isotropic materials, the inelastic and thermomechanical analysis of components is now being performed but is still under development. For the case of anisotropic materials such as single crystal superalloys, directionally solidified alloys and metal matrix composites, anisotropy is added to the list of complicated phenomena which must be modeled. All of these analytical methods are inherently nonlinear and require numerical methods for their implementation. Fortunately, the growths of computing power and economy make possible nonlinear analyses in the design of critical and severely stressed components.

The isothermal methods of anisotropic inelastic analysis range from yield surface approaches (Lee and Zaverl, 1978; Eisenberg and Yen, 1981, 1984) to phenomenological formulations using "unified" theories (Stouffer and Bodner, 1979; Robinson, 1983; Dame, 1985; Sutcu, 1985; Sutcu and Krempl, 1986, 1990; Walker and Jordan, 1989). Dame (1985) and Walker and Jordan (1989) used a "crystallographic" approach where the constitutive equation of each slip system in a cubic crystal was given by the "unified" constitutive equation of the respective authors. Following the classical methods of single crystal plasticity the overall response was obtained by summing up the contributions of each slip system.

The present infinitesimal thermomechanical theory is intended for application to anisotropic engineering alloys for high temperature service in the power generation and propulsion industries. The intention is to present a theory which can model essential macroscopic phenomena with the least possible complexity. The behavior of these alloys, which

† Now with Lucky Ltd, Dae-deog Danji, Dae-jeon, Korea.

include single crystals, directionally solidified alloys and metal matrix composites, is often very complex. As an example, some alloys may exhibit a strength increase over a limited temperature range or may show tension/compression asymmetry. As a consequence the formulation does not start from the notion of an Arrhenius relation between temperature and strain rate. Any temperature dependence is allowed.

In theoretical treatments, the use of a thermodynamic potential provides consistency and conciseness and many known constitutive equations can be derived. When new material laws are the aim, the thermodynamic approach is not directly useful as the means of experimentation provide measures for stress and strain but not for potentials. Also, as a minimum, the potential and the growth laws for the state variables have to be postulated before the mathematical formalism can be employed. In the present approach, the constitutive equation for the inelastic strain rate (the flow law), and the growth laws for the state variables are postulated. The number of postulates is the same in both cases. After the stress–strain laws are established, the thermodynamic potentials can be constructed.

Rather than using representation theorems and coordinate-free notation (Sutcu, 1985; Sutcu and Krempl, 1986), material matrices are employed. Representation with respect to the preferred material axis is assumed. For orthotropy, the matrices are symmetric and have nine independent constants. Elastic and inelastic Poisson's ratio matrices which depend only on temperature are introduced in the generalization of the previously proposed isotropic viscoplasticity based on overstress (VBO) (Yao and Krempl, 1985; Krempl *et al.*, 1986; Krempl, 1987). Compressible and incompressible inelastic deformations are considered. The orthotropic theory contains transverse isotropy, cubic symmetry and isotropy as special cases.

ORTHOTROPIC THEORY OF VISCOPLASTICITY BASED ON OVERSTRESS FOR THERMOMECHANICAL DEFORMATION (TVBO)

For the representation of the equations, vector notation is used where stress tensor components σ_{ij} and the small strain tensor components ε_{ij} are related to their vector components by

$$\sigma_1 = \sigma_{11},\quad \sigma_2 = \sigma_{22},\quad \sigma_3 = \sigma_{33},\quad \sigma_4 = \sigma_{23},\quad \sigma_5 = \sigma_{31},\quad \sigma_6 = \sigma_{12},$$

and

$$\varepsilon_1 = \varepsilon_{11},\quad \varepsilon_2 = \varepsilon_{22},\quad \varepsilon_3 = \varepsilon_{33},\quad \varepsilon_4 = 2\varepsilon_{23},\quad \varepsilon_5 = 2\varepsilon_{31},\quad \varepsilon_6 = 2\varepsilon_{12},$$

respectively.

Capital boldface letters denote 6×6 matrices. The components of these matrices are given with respect to the on-axis, xyz Cartesian coordinate system which coincides with the material symmetry axes, or with respect to an off-axis, 123 coordinate system. The components are identified by the respective subscripts x, y, z or 1, 2, 3. An orthotropic matrix is symmetric and has nine independent components, see eqn (1.12) of Christensen (1979).

Vectors and matrices are primed and unprimed when referring to the 123-system and xyz-system, respectively. All constants and functions can depend on $T - T_0$, where T is the absolute temperature and T_0 is a reference temperature. The temperature dependence is not specially displayed.

The formulation given below represents cyclic neutral behavior and does not include recovery of state and aging. As a consequence two tensorial state variables and their growth laws are needed. The modeling of cyclic hardening/softening requires additional state variables. These extensions will be proposed in a future paper.

The flow law

In the context of an infinitesimal theory, the total strain rate, $d\varepsilon'/dt$, is considered to be the sum of elastic, $d\varepsilon'^{el}/dt$, inelastic, $d\varepsilon'^{in}/dt$, and thermal strain rates, $d\varepsilon'^{th}/dt$,

$$\dot{\varepsilon}' = \dot{\varepsilon}'^{el} + \dot{\varepsilon}'^{in} + \dot{\varepsilon}'^{th}, \tag{1}$$

where the sum of the elastic and inelastic strain rate is called the mechanical strain rate,

$$\dot{\varepsilon}'^{me} = \dot{\varepsilon}'^{el} + \dot{\varepsilon}'^{in}. \tag{2}$$

A superposed dot represents the total time derivative, d/dt.

For each strain rate, a constitutive equation is postulated. The elastic strain is assumed to be independent of thermal history, therefore,

$$\dot{\varepsilon}'^{el} = \frac{d}{dt}(\mathbf{N}^{-1}\mathbf{C}^{-1}\mathbf{M}\sigma'). \tag{3}$$

A superscript -1 denotes the inverse of a matrix. As in the case of isotropy, the inelastic strain rate is only a function of the overstress $\mathbf{x}$. It denotes the difference between the stress σ and the equilibrium stress $\mathbf{g}$, a state variable of the theory,

$$\dot{\varepsilon}'^{in} = \mathbf{N}^{-1}\mathbf{K}^{-1}\mathbf{M}\mathbf{x}'. \tag{4}$$

The thermal strain rate is given by

$$\dot{\varepsilon}'^{th} = \mathbf{N}^{-1}\alpha\dot{T}. \tag{5}$$

In the above, $\mathbf{N}$ and $\mathbf{M}$ are the transformation matrices for strain and stress, respectively. They are given as

$$\varepsilon = \mathbf{N}\varepsilon', \tag{6}$$

$$\sigma = \mathbf{M}\sigma'. \tag{7}$$

The elastic modulus matrix $\mathbf{C}$ is symmetric. Its components can depend on temperature. It is written as a product of the diagonal elastic modulus matrix $\mathbf{C}_d$ and the matrix $\mathbf{R}_e$ of the elastic Poisson's ratios. The components of the first are the elastic moduli, those of the latter are only related to Poisson's ratios. We have

$$\mathbf{C} = \mathbf{C}_d\mathbf{R}_e, \quad \mathbf{C}^t = \mathbf{C}; \tag{8}$$

a superscript t denoting transpose of a matrix

where

$$\mathbf{C}_d = \begin{bmatrix} E_{xx} & & & & & \\ & E_{yy} & & & & \\ & & E_{zz} & & & \\ & & & G_{yz} & & \\ & & & & G_{zx} & \\ & & & & & G_{xy} \end{bmatrix} \tag{9}$$

and

$$
\mathbf{R}_c = \begin{bmatrix}
\frac{1-\nu_{yz}\nu_{zy}}{\nu_0} & \frac{\nu_{xy}+\nu_{xz}\nu_{zy}}{\nu_0} & \frac{\nu_{xz}+\nu_{xy}\nu_{yz}}{\nu_0} & & & \\
\frac{\nu_{yx}+\nu_{zx}\nu_{yz}}{\nu_0} & \frac{1-\nu_{xz}\nu_{zx}}{\nu_0} & \frac{\nu_{yz}+\nu_{yx}\nu_{xz}}{\nu_0} & & & \\
\frac{\nu_{zx}+\nu_{yx}\nu_{zy}}{\nu_0} & \frac{\nu_{zy}+\nu_{xy}\nu_{zx}}{\nu_0} & \frac{1-\nu_{xy}\nu_{yx}}{\nu_0} & & & \\
 & & & 1 & & \\
 & & & & 1 & \\
 & & & & & 1
\end{bmatrix} \tag{10}
$$

with

$$
\nu_0 = 1-\nu_{xz}\nu_{zx}-\nu_{yz}\nu_{zy}-\nu_{xy}\nu_{yx}-\nu_{xy}\nu_{yz}\nu_{zx}-\nu_{yx}\nu_{xz}\nu_{zy}\,;
$$

$$
\nu_{ij} = -\frac{\dot{\varepsilon}_i^{\rm el}}{\dot{\varepsilon}_j^{\rm el}} \quad (i,j = x,y,z, \quad i \neq j) \text{ the elastic Poisson's ratios}
$$

for uniaxial loading in the j direction. They may only depend on temperature. The symmetry of $\mathbf{C}$ requires the reciprocity relations $\nu_{ji}/E_{ii} = \nu_{ij}/E_{jj}$ (no sum, $i,j = x,y,z, i \neq j$). [For the form of the inverse $\mathbf{R}_c^{-1}$, see (37).]

Similarly, the vicosity matrix $\mathbf{K}$ is given by

$$
\mathbf{K} = \mathbf{K}_d\mathbf{R}_k, \quad \mathbf{K}' = \mathbf{K} \tag{11}
$$

where

$$
\mathbf{K}_d = \begin{bmatrix}
K_{xx} & & & & & \\
 & K_{yy} & & & & \\
 & & K_{zz} & & & \\
 & & & K_{yz} & & \\
 & & & & K_{zx} & \\
 & & & & & K_{xy}
\end{bmatrix} k[\Gamma], \tag{12}
$$

(square brackets enclosing a symbol denote "function of") and

$$
\mathbf{R}_k = \begin{bmatrix}
\frac{1-\eta_{yz}\eta_{zy}}{\eta_0} & \frac{\eta_{xy}+\eta_{xz}\eta_{zy}}{\eta_0} & \frac{\eta_{xz}+\eta_{xy}\eta_{yz}}{\eta_0} & & & \\
\frac{\eta_{yx}+\eta_{zx}\eta_{yz}}{\eta_0} & \frac{1-\eta_{xz}\eta_{zx}}{\eta_0} & \frac{\eta_{yz}+\eta_{yx}\eta_{xz}}{\eta_0} & & & \\
\frac{\eta_{zx}+\eta_{yx}\eta_{zy}}{\eta_0} & \frac{\eta_{zy}+\eta_{xy}\eta_{zx}}{\eta_0} & \frac{1-\eta_{xy}\eta_{yx}}{\eta_0} & & & \\
 & & & 1 & & \\
 & & & & 1 & \\
 & & & & & 1
\end{bmatrix} \tag{13}
$$

with

$$
\eta_0 = 1-\eta_{xz}\eta_{zx}-\eta_{yz}\eta_{zy}-\eta_{xy}\eta_{yx}-\eta_{xy}\eta_{yz}\eta_{zx}-\eta_{yx}\eta_{xz}\eta_{zy}\,;
$$

$$
\eta_{ij} = -\frac{\dot{\varepsilon}_i^{\rm in}}{\dot{\varepsilon}_j^{\rm in}} \quad (i,j = x,y,z, \quad i \neq j) \text{ the inelastic Poisson's ratios}
$$

for uniaxial loading in the j direction. They may only depend on temperature. The function $k[\Gamma]$ has the dimension of stress × time and is the viscosity function of Krempl *et al.* (1986). It is positive and decreasing and must be determined for a material. Specific functions that fulfil the above requirements can be found in Krempl (1982) and Yao and Krempl (1985). The symmetry of **K** requires $\eta_{ji}/K_{ii} = \eta_{ij}/K_{jj}$ (no sum, $i, j = x, y, z, i \neq j$) and allows only a common functional dependence of **K** through $k[\Gamma]$, see (12).

The orthotropic invariant Γ is given by Krempl and Hong (1989) and Sutcu and Krempl (1990)

$$\Gamma^2 = |\mathbf{x}^t\mathbf{H}\mathbf{x} + \mathbf{a}^t\mathbf{x}|, \tag{14}$$

where $\mathbf{x} = \boldsymbol{\sigma} - \mathbf{g}$ is the overstress, **H** is a dimensionless orthotropic matrix and the vector $\mathbf{a}^t = [a_x \quad a_y \quad a_z \quad 0 \quad 0 \quad 0]$, dimension of stress, is the repository for viscous tension/compression asymmetry (see Sutcu and Krempl, 1990). (If **a** is set to zero, viscous tension/compression symmetry results.) Again **H** and **a** are, like the elastic moduli, material properties and must be determined for a given material.

The vector $\boldsymbol{\alpha}^t = [\alpha_x \quad \alpha_y \quad \alpha_z \quad 0 \quad 0 \quad 0]$ denotes the coefficient of thermal expansion.

Growth laws for the state variables

Formulation. The growth law for $\mathbf{g}'$ is the repository for modeling elastic regions and hysteresis. It is given by

$$\dot{\mathbf{g}}' = \mathbf{M}^{-1}\mathbf{B}_d\mathbf{C}_d^{-1}\mathbf{M}\dot{\boldsymbol{\sigma}}' + \dot{T}\mathbf{M}^{-1}\frac{\partial}{\partial T}(\mathbf{B}_d\mathbf{C}_d^{-1})\mathbf{M}\boldsymbol{\sigma}' + \mathbf{M}^{-1}(\mathbf{B}_d^* - \mathbf{D}_d)\mathbf{K}_d^{-1}\mathbf{M}\mathbf{x}'. \tag{15}$$

The first and the last terms represent the elastic and the inelastic contributions to the growth of **g**, respectively. The term multiplied by $\partial T/\partial t$ ensures independence of thermal history of the elastic growth of **g** (see Lee and Krempl, 1990b).

The newly introduced quantities are defined as follows:

The diagonal shape function matrix $\mathbf{B}_d$ which controls the elastic growth is

$$\mathbf{B}_d = \begin{bmatrix} \psi_{xx}[\Gamma] & & & & & \\ & \psi_{yy}[\Gamma] & & & & \\ & & \psi_{zz}[\Gamma] & & & \\ & & & \psi_{yz}[\Gamma] & & \\ & & & & \psi_{zx}[\Gamma] & \\ & & & & & \psi_{xy}[\Gamma] \end{bmatrix}; \tag{16a}$$

the diagonal shape function matrix $\mathbf{B}_d^*$ associated with inelastic growth is

$$\mathbf{B}_d^* = \begin{bmatrix} \phi_{xx}[\Gamma] & & & & & \\ & \phi_{yy}[\Gamma] & & & & \\ & & \phi_{zz}[\Gamma] & & & \\ & & & \phi_{yz}[\Gamma] & & \\ & & & & \phi_{zx}[\Gamma] & \\ & & & & & \phi_{xy}[\Gamma] \end{bmatrix}. \tag{16b}$$

The functions $\psi_{ij}[\Gamma]$ and $\phi_{ij}[\Gamma]$ are positive and nonincreasing. They are patterned after the shape functions introduced by Krempl and Yao (1985) and Krempl *et al.* (1986). [Instead of making ψ_{ij} and ϕ_{ij} functions of Γ, they could alternatively be functions of Θ defined in (19) below.] The matrix $\mathbf{D}_d$ is

$$\mathbf{D_d} = \Theta(\mathbf{B_d^*} - \mathbf{H_d}(\mathbf{I} - \mathbf{B_d C_d^{-1}})) \tag{17}$$

with

$$\mathbf{H_d} = \begin{bmatrix} E_{txx} & & & & & \\ & E_{tyy} & & & & \\ & & E_{tzz} & & & \\ & & & E_{tyz} & & \\ & & & & E_{tzx} & \\ & & & & & E_{txy} \end{bmatrix}, \tag{18}$$

where the E_{tii} are the tangent moduli of the stress–inelastic strain curves at the maximum strain of interest. They can be positive, zero or negative. The invariant Θ is defined as (see Stucu and Krempl, 1990)

$$\Theta^2 = |\mathbf{z}^t \mathbf{P} \mathbf{z} + \mathbf{b}^t \mathbf{z}|, \tag{19}$$

where **P** is an orthotropic matrix whose component have the dimension of reciprocal stress squared and the vector $\mathbf{b}^t = [b_x \quad b_y \quad b_z \quad 0 \quad 0 \quad 0]$, with dimension of reciprocal stress, is the repository for tension/compression asymmetry of the time-independent contribution to the stress, **z** (see Sutcu and Krempl, 1990). The vector **z** is given as

$$\mathbf{z} = \mathbf{g} - \mathbf{f} \tag{20}$$

and

$$\dot{\mathbf{f}}' = \mathbf{M}^{-1} \mathbf{H_d} \mathbf{K_d}^{-1} \mathbf{M} \dot{\mathbf{x}}'. \tag{21}$$

Asymptotic analyses for the uniaxial isothermal case by Krempl *et al.* (1986) and Sutcu and Krempl (1989) show that $d\mathbf{f}/dt$ determines $d\boldsymbol{\sigma}/dt$ ultimately. The purpose of (21) is to set this slope. When tangent moduli E_{tii} are equal to zero, stress–strain curves become ultimately horizontal. In this paper the E_{tii} have to be interpreted as the slopes in a stress–plastic strain diagram at the maximum plastic strain. Substitution of (4) into (21) reveals a form similar to the Prager–Ziegler kinematic hardening law. Therefore **f** is called the kinematic variable or kinematic stress.

Restrictions on the growth laws for the state variables. In the limit as all rates go to zero, eqns (1)–(4) show that the stress equals the equilibrium stress. It has also been shown for the isotropic, isothermal case that the stress approaches the equilibrium stress as the total strain rate goes to zero in the limit (Cernocky and Krempl, 1979; Krempl *et al.*, 1986). These facts suggest that whenever the boundary conditions require zero stress components, the corresponding equilibrium stress components with the initial value zero, must at all times also be zero. (It is also possible to require that the equilibrium stress components corresponding to the zero stress components reduce to zero only in the limit as the rates go to zero. This approach will not be pursued here.) These conditions must hold for arbitrary **M** and **N**. Therefore the following restrictions must be imposed on the matrices which appear in (15) and (21). **I** is the 6×6 identity matrix.

$$\mathbf{B_d C_d^{-1}} = q_1 \mathbf{I}, \tag{22}$$

$$\mathbf{H_d K_d^{-1}} = (p/k[\Gamma])\mathbf{I}, \tag{23}$$

$$\mathbf{B}_d^*\mathbf{K}_d^{-1} = \frac{q_2}{k[\Gamma]}\mathbf{I}, \tag{24}$$

where

$$q_1 = \frac{\psi_{xx}[\Gamma]}{E_{xx}} = \frac{\psi_{yy}[\Gamma]}{E_{yy}} = \frac{\psi_{zz}[\Gamma]}{E_{zz}} = \frac{\psi_{xy}[\Gamma]}{G_{xy}} = \frac{\psi_{yz}[\Gamma]}{G_{yz}} = \frac{\psi_{zx}[\Gamma]}{G_{zx}}. \tag{25}$$

$$q_2 = \frac{\phi_{xx}[\Gamma]}{K_{xx}} = \frac{\phi_{yy}[\Gamma]}{K_{yy}} = \frac{\phi_{zz}[\Gamma]}{K_{zz}} = \frac{\phi_{xy}[\Gamma]}{K_{xy}} = \frac{\phi_{yz}[\Gamma]}{K_{yz}} = \frac{\phi_{zx}[\Gamma]}{K_{zx}}, \tag{26}$$

$$p = \frac{E_{txx}}{K_{xx}} = \frac{E_{tyy}}{K_{yy}} = \frac{E_{tzz}}{K_{zz}} = \frac{E_{txy}}{K_{xy}} = \frac{E_{tyz}}{K_{yz}} = \frac{E_{tzx}}{K_{zx}}. \tag{27}$$

The viscosity factors, K_{ij}, have dimension stress and $k[\Gamma]$ has the dimension time. The dimensionless factors q_1 and q_2 are called modified shape functions; they control the shape of the stress–strain diagram. When these conditions are implemented, the growth laws of the state variables are reduced to

$$\dot{\mathbf{g}}' = q_1\dot{\boldsymbol{\sigma}}' + \dot{T}\frac{\partial q_1}{\partial T}\boldsymbol{\sigma}' + (q_2 - \Theta(q_2 - p(1-q_1)))\frac{\mathbf{x}'}{k[\Gamma]}, \tag{28}$$

$$\dot{\mathbf{f}}' = \frac{p}{k[\Gamma]}\mathbf{x}'. \tag{29}$$

These equations replace, in consecutive order, the more complicated equations (15) and (21). All other equations remain unchanged.

If the unrestricted equations (15) and (21) were to be used, then equilibrium stress components could develop when there are no corresponding stress components. The "extra" equilibrium components could cause the numerical solution to become unstable and could lead to unexpected and unrealistic results. Examples are given by Sutcu (1985).

It should be noted that orthotropy is maintained with (28) and (29). The orthotropic invariants Γ and Θ are unrestricted and they enter nonlinearly into the equations. Also the viscosity factors K_{ij} can be chosen independently of the elastic moduli when $q_1 \neq q_2$.

The growth law for the equilibrium stress looks somewhat unfamiliar since it is written in terms of stress rate and overstress. Substitution of (3) and (4) into (28) and (29), using (25)–(27), yields the familiar forms in terms of strain rates,

$$\dot{\mathbf{g}}' = q_1\mathbf{M}^{-1}\mathbf{C}_d\mathbf{R}_c\mathbf{N}\dot{\boldsymbol{\varepsilon}}'^{el} + (q_1 - \Theta(q_2 - p(1-q_1)))\mathbf{M}^{-1}\mathbf{K}_d\mathbf{R}_k\mathbf{N}\dot{\boldsymbol{\varepsilon}}'^{in}$$
$$+ \dot{T}\mathbf{M}^{-1}\left(q_1\frac{\partial \mathbf{C}_d}{\partial T}\mathbf{R}_c + q_1\mathbf{C}_d\frac{\partial \mathbf{R}_c}{\partial T} + \frac{\partial q_1}{\partial T}\mathbf{C}_d\mathbf{R}_c\right)\mathbf{N}\boldsymbol{\varepsilon}'^{el}, \tag{30}$$

$$\dot{\mathbf{f}}' = p\mathbf{M}^{-1}\mathbf{K}_d\mathbf{R}_k\mathbf{N}\dot{\boldsymbol{\varepsilon}}'^{in}. \tag{31}$$

The anisotropic nature of (30) and (31) is apparent. We have found it useful and easy to work with (28) and (29) rather than with the familiar but complicated equations (30) and (31).

Reduction to plane stress/strain

Plane stress. Let σ_3, σ_4 and σ_5 be the zero stress components and let the z-axis coincide with the 3-axis. If the initial conditions of the stress and the equilibrium stress components with indices 3, 4 and 5 are zero, then they will always stay zero on account of (28). The same argument can be made for the components of **f**. Therefore the stress, strain, and state variable vectors are given by

$$\begin{aligned} \boldsymbol{\varepsilon}'^{\mathrm{t}} &= [\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_6], \\ \boldsymbol{\sigma}'^{\mathrm{t}} &= [\sigma_1 \quad \sigma_2 \quad \sigma_6], \\ \mathbf{g}'^{\mathrm{t}} &= [g_1 \quad g_2 \quad g_6], \\ \mathbf{f}'^{\mathrm{t}} &= [f_1 \quad f_2 \quad f_6]. \end{aligned} \tag{32}$$

The out-of-plane component of the strain is entirely determined by in-plane components and is given by

$$\dot{\varepsilon}_3 = \dot{\varepsilon}_z = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_{zx}}{E_{xx}}\sigma_x + \frac{v_{zy}}{E_{yy}}\sigma_y\right) - \frac{1}{k[\Gamma]}\left(\frac{\eta_{zx}x_x}{K_{xx}} + \frac{\eta_{zy}x_y}{K_{yy}}\right) + \alpha_z\dot{T}. \tag{33}$$

Plane strain. In this case ε_3, ε_4 and ε_5 are set equal to zero. If the initial conditions of the stress and the equilibrium stress with indices 4 and 5 are zero, then these quantities and the kinematic stress will remain zero. The out-of-plane components of the stress, equilibrium stress and kinematic stress develop and contribute to the invariants Γ and Θ.

DISCUSSION

General remarks

This orthotropic TVBO not only applies to the case of variable temperature but generalizes the simplified isothermal version with constant Poisson's ratio, presented by Sutcu and Krempl (1990). It will be shown below that a variable Poisson's ratio can be represented with the present theory. Setting aside the thermal aspects, the present theory differs from that given by Sutcu and Krempl (1990) by the introduction of the inelastic Poisson's ratio matrix $\mathbf{R}_k$ and by the formulation of the growth laws for the equilibrium stress **g** and the kinematic variable **f** in terms of stress and overstress as given by (28) and (29). This formulation is advantageous in numerical calculations and in manipulations. This theory retains all the properties demonstrated by Sutcu and Krempl (1990) which includes tension/compression asymmetry, quasi-elastic behavior in one direction and visco-plastic behavior in the other directions, orthotropic strain rate sensitivity, creep, and relaxation. Recovery of state and aging as well as cyclic hardening/softening are not part of the present theory but will be incorporated in the future.

Since the formulation of the present theory assumes that the material constants are functions of current temperature only, the asymptotic rate response and the ultimate level of $\mathbf{g}-\mathbf{f}$ are independent of temperature history. For any history which ends up with the same temperature and the same mechanical strain rate, the model predicts ultimately the same stress rate response. Such behavior is reported by Chan and Lindholm (1990). If metallurgical changes, such as phase changes and strain aging, occur, independence of thermal history may no longer be an adequate assumption and the model will have to be modified to account for thermal history effects (see Lee and Krempl, 1990b). Independence of thermal history implies that the constants and functions of the theory can be determined from isothermal tests at a sequence of constant temperatures and then applied to thermal loading within these temperature ranges. The partial derivatives with respect to temperature in (15), (28) and (30), which are introduced to model elastic deformations independent of thermal history, will influence the transient and asymptotic behavior under temperature changes. These are introduced in response to recent observations in the modeling of hysteresis under thermal and mechanical cycling (Walker, 1981; Moreno and Jordan, 1986; Chaboche, 1987). The implications of these terms in modeling thermomechanical behavior are investigated systematically by Lee and Krempl (1990b).

The present theory involves two modified shape functions q_1 and q_2 and the growth law for the kinematic variable **f** has only inelastic growth. But the isotropic theory proposed by Yao and Krempl (1985) and Krempl and Yao (1987) has one shape function and the growth law for the kinematic variable f has both elastic and inelastic growth. When $q_1 = q_2$ and $K_{ij} = E_{ij}$, the elastic moduli and tangent moduli at the maximum strain of interest are related by (27). In addition, reduction to the isotropic case makes the elastic Poisson's ratio equal to the inelastic Poisson's ratio. Introducing q_1 and q_2 avoids this restriction. The purpose of f is to set the slope at the maximum strain of interest; (29) or (31) accomplish this goal and are simpler than the growth laws previously used. This new formulation was also adopted by Nishiguchi *et al.* (1990).

Variable Poisson's ratio

In analogy to the definitions following (10) and (13), the actual variable Poisson's ratio based on strain rates is introduced as

$$\gamma_{ij} = -\frac{\dot{\varepsilon}_i^{me}}{\dot{\varepsilon}_j^{me}}, \quad i,j = x,y,z, \quad i \neq j \tag{34}$$

for the uniaxial loading in the j direction. From eqn (2) and the definitions of the elastic and inelastic Poisson's ratios, following (10) and (13), respectively, the actual Poisson's ratio can be written in terms of the elastic and inelastic Poisson's ratios,

$$\gamma_{ij}\dot{\varepsilon}_j^{me} = \frac{\mathrm{d}}{\mathrm{d}t}(v_{ij}\varepsilon_j^{el}) + \eta_{ij}\dot{\varepsilon}_j^{in}, \quad \text{no sum on } j, \quad i,j = x,y,z, \quad i \neq j. \tag{35}$$

Since the loading is uniaxial, (35) can be expressed in terms of stress rate and overstress. From (2), (3), (4), (12) and (35)

$$\gamma_{ij}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sigma_j}{E_{jj}}\right) + \frac{x_j}{K_{jj}k[\Gamma]}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(v_{ij}\frac{\sigma_j}{E_{jj}}\right) + \eta_{ij}\frac{x_j}{K_{jj}k[\Gamma]} \tag{36}$$

(no sum on j, $i,j = x,y,z, i \neq j$). When the elastic and the inelastic Poisson's ratios are given, (36) permits the calculation of the actual Poisson's ratio.

For multiaxial loading the actual variable Poisson's ratio matrix based on rates, $\mathbf{R}_m$, can be written as

$$\mathbf{R}_m^{-1} = \begin{bmatrix} 1 & -\gamma_{xy} & -\gamma_{xz} & & & \\ -\gamma_{yx} & 1 & -\gamma_{yz} & & & \\ -\gamma_{zx} & -\gamma_{zy} & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \tag{37}$$

and can be used to construct the mechanical strain rate vector

$$\dot{\boldsymbol{\varepsilon}}^{me} = \mathbf{R}_m^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{C}_d^{-1}\boldsymbol{\sigma}) + \mathbf{K}_d^{-1}\mathbf{x}\right). \tag{38}$$

From (2), (3), (4), (10), (13), (37) and (38), a relation between $\mathbf{R}_m$, $\mathbf{R}_e$ and $\mathbf{R}_k$ can be inferred,

$$\mathbf{R}_m^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{C}_d^{-1}\boldsymbol{\sigma}) + \mathbf{K}_d^{-1}\mathbf{x}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{R}_e^{-1}\mathbf{C}_d^{-1}\boldsymbol{\sigma}) + \mathbf{R}_k^{-1}\mathbf{K}_d^{-1}\mathbf{x}. \tag{39}$$

A constant Poisson's ratio was assumed by Sutcu and Krempl (1990). This possibility is contained in the present theory. For temperature-independent material properties and $\nu_{ij} = \eta_{ij}$, the Poisson's ratio γ_{ij} determined from either (36) or (39) is equal to ν_{ij}.

Inelastic incompressibility, invariance of inelastic deformation under superposed pressure, tension/compression symmetry

In the derivation of isotropic theories of rate-independent plasticity, inelastic incompressibility and invariance under superposed pressure are used interchangeably to formulate the inelastic strain as a function of the stress deviator and its invariants. When only quadratic invariants are used, tension/compression symmetry follows.

Each of these conditions leads to different reductions when applied to the present orthotropic theory.

Inelastic incompressibility. In plasticity theory it is generally assumed that the inelastic strains are volume preserving. So far this condition has not been used. Inelastic incompressibility requires

$$\dot{\varepsilon}_x^{in} + \dot{\varepsilon}_y^{in} + \dot{\varepsilon}_z^{in} = 0. \tag{40}$$

From (4) and (40), and since x_x, x_y and x_z are arbitrary, the inelastic incompressibility condition is uniquely satisfied by setting

$$\begin{aligned} \eta_{yx} + \eta_{zx} &= 1, \\ \eta_{xy} + \eta_{zy} &= 1, \\ \eta_{xz} + \eta_{yz} &= 1. \end{aligned} \tag{41}$$

Since **K** is symmetric [see (11)], (41) can be expressed as

$$\begin{aligned} \eta_{xy} &= 0.5\left(1 - \frac{K_{yy}}{K_{zz}} + \frac{K_{yy}}{K_{xx}}\right), \\ \eta_{yz} &= 0.5\left(1 - \frac{K_{zz}}{K_{xx}} + \frac{K_{zz}}{K_{yy}}\right), \\ \eta_{zx} &= 0.5\left(1 - \frac{K_{xx}}{K_{yy}} + \frac{K_{xx}}{K_{zz}}\right). \end{aligned} \tag{42}$$

For the case of isotropy or of cubic symmetry ($K_{xx} = K_{yy} = K_{zz}$), the inelastic Poisson's ratios are equal to 0.5 as they should be. By substituting either (41) or (42) into (4), an orthotropic theory for inelastic incompressibility is obtained. Note, however, that the growth laws (28) and (29) must be used. The matrix $\mathbf{R}_k$ does not exist and (30) and (31) are not useful.

Invariance of inelastic deformation under superposed pressure. Invariance of the inelastic deformation under superposed pressure requires that (41) or (42) hold in addition to

$$\begin{aligned} H_{xx} + H_{xy} + H_{xz} &= 0, \\ H_{xy} + H_{yy} + H_{yz} &= 0, \\ H_{xz} + H_{yz} + H_{zz} &= 0 \end{aligned} \tag{43}$$

and

$$a_x + a_y + a_z = 0, \tag{44}$$

as well as similar conditions on **P** and **b**. Note that invariance under superposed pressure requires inelastic incompressibility but does not imply tension/compression symmetry.

Tension/compression symmetry. By setting $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$, tension/compression symmetry can be modeled. This condition can be applied whether or not inelastic incompressibility or invariance of inelastic deformation under superposed pressure is represented.

Reduction to isotropy

In this case all orthotropic matrices are replaced by the corresponding isotropic ones as shown in the Appendix. There are two independent elastic moduli, two independent inelastic constants (the viscosity function and the inelastic Poisson's ratio), two shape functions (ψ could be set equal to ϕ), one tangent modulus, two isotropic invariants, and the coefficient of thermal expansion. The model consists now of (1), (3), (4), (5), (14), (19), and (28), (29) with all the orthotropic matrices replaced by the corresponding isotropic ones.

To show that the isotropic model derived from the orthotropic version corresponds to the one initially proposed by Yao and Krempl (1985), the isotropic version is written in terms of deviatoric and hydrostatic components in the Appendix. In this version the condition of inelastic incompressibility has not yet been invoked and therefore all deviatoric and hydrostatic components can be calculated.

Inelastic incompressibility. Setting $\eta = 0.5$ in (A8) invokes the inelastic incompressibility condition and renders x_h indeterminate in (A7). However, the growth laws for the hydrostatic component of **g** and $\boldsymbol{\sigma}$ in (A10) and (A6), respectively, permit the determination of x_h. With x_h known, f_h can be calculated from (A12). The present theory permits the calculation of all hydrostatic components in the presence of inelastic incompressibility. This is accomplished by the "stress formulation" of the growth laws for **g** and **f** adopted in (28) and (29), respectively.

Even if inelastic incompressibility is assumed by setting $\eta = 0.5$, the model can still predict a superposed hydrostatic pressure effect for inelastic deformation and tension/compression asymmetry through the invariants Γ and Θ. Generally, it is assumed that inelastic incompressibility or tension/compression symmetry are synonymous.

Inelastic invariance under superposed pressure. To have the inelastic strain invariant under superposed pressure, we have to require that $\eta = 0.5$, $a = 0$, $H_2 = 3H_1$, $b = 0$ and $P_2 = 3P_1$. In this case inelastic incompressibility and tension/compression symmetry follow. With these stipulations, the invariants Γ and Θ become second invariants of deviators.

Tension/compression symmetry. To model tension/compression symmetry of inelastic deformation alone, $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$ are required.

Relation to previously proposed isotropic theory. When invariance under superposed pressure is invoked (which results in tension/compression symmetry), the equations of the Appendix reduce to the theory presented by Yao and Krempl (1985) and Krempl and Yao (1987) with the exception of the growth laws for **g** and **f**. Yao and Krempl (1985) write eqns (1)-(10) without a distinction between elastic and inelastic Poisson's ratios. Poisson's ratio was set equal to 0.5 after the equations had been reduced to component forms. The indeterminacy of eqn (15) of Yao and Krempl (1985) for this case was therefore not realized. The equations given by Krempl and Yao (1987) distinguish between elastic and inelastic Poisson's ratios in the flow law, but use the value 0.5 for the growth laws of **g** and **f**, see (1, 2) of Krempl and Yao (1987).

The difference between the present formulation and the one of Krempl and Yao (1987) is evident by examining (A14) and (A15). By setting $\eta = \nu = 0.5$ in (A14), eqn (A15) is obtained except for the direction of the last term. It is in the direction of the inelastic strain rate in the present theory, but in the direction of the difference between the deviators of the equilibrium stress and the kinematic stress in Krempl and Yao (1987). The formulation similar to (A14) was introduced by Sutcu (1985) for mathematical convenience after it had been shown in some numerical experiments for the uniaxial case that the two formulations in Sutcu (1985) and Krempl and Yao (1987) differed insignificantly. The differences for multiaxial loading need to be explored. It is interesting to note that a similar modification was introduced by Burlet and Cailletaud (1987) in the context of the growth law for the backstress in a rate-independent formulation.

The consistent formulation in terms of elastic and inelastic Poisson's ratios which leads to (A14) was independently proposed by Nishiguchi (see Nishiguchi *et al.*, 1990).

Modeling of real material behavior

Papers by Krempl and Hong (1989) and Lee and Krempl (1988, 1990a) deal with the numerical simulation of metal matrix composites under isothermal and variable temperature conditions. Therefore no applications are given here.

The present theory generalizes the theory introduced in Krempl and Hong (1989) to the cases of variable Poisson's ratio and temperature. It is shown in Krempl and Hong (1989) that the on- and off-axis behavior of metal matrix plies can be reproduced under monotonic and cyclic loading.

Thermomechanical loading is treated in the context of a simple laminate theory by Lee and Krempl (1988, 1990a) using TVBO. The on- and off-axis behavior of plies and the residual stresses in laminates made of metal matrix composites are calculated for thermomechanical loading.

TVBO is specialized by Choi and Krempl (1989) for cubic symmetry to simulate the behavior of single crystal superalloys when loaded in the cube side-, body diagonal-, and face diagonal-directions under isothermal conditions. The results are promising.

Possible simplification and relation to plasticity

In the present theory, the matrices **H**, **P** and the vectors **a**, **b** in (14) or (19) have to be independently selected. Following Sutcu (1985), Sham (1989) suggested replacing Γ in (14) by

$$\Gamma^2 = (\mathbf{K}^{-1}\mathbf{x})^t(\mathbf{K}^{-1}\mathbf{x}), \tag{45}$$

with an analogous definition of Θ instead of (19). Such a formulation would considerably reduce the constants needed. Obviously the modeling of tension/compression asymmetry through the invariants would be lost. However, a further possibility exists to model tension/compression asymmetry through the initial condition of **f**, in (31) (Lee, 1989).

In the present theory, all matrix components are constants which can only depend on temperature. The matrices do not change with the state of stress; only the invariants Γ and Θ do, together with the stress and equilibrium stress vectors.

In rate-independent isotropic plasticity, the flow rule (tensor notation is used here)

$$\dot{\varepsilon}^{p}_{ij} = \lambda \frac{\partial f}{\partial s_{ij}}, \tag{46}$$

can be rewritten as (see Yamada *et al.*, 1968; Dvorak and Bahei-El-Din, 1982)

$$\dot{\varepsilon}^{p}_{ij} = B M_{ijkl}\dot{s}_{kl}, \tag{47}$$

with B a scalar expression and $M_{ijkl} = (\partial f/\partial s_{ij})s_{kl}$. With the usual assumption of a quadratic yield surface $M_{ijkl} = s_{ij}s_{kl}$. Thus the entries of the matrix **M** depend on the state of stress. Specifically, when axial and shear stresses are present, shear stress increments can cause normal plastic strain increments.

The equivalent of **M** in the TVBO, the matrix $\mathbf{K}^{-1}$ [see (11)] has constant components. However, it can be seen from (1)–(4) that shear stress increments can cause normal and shear inelastic strains when shear and normal stresses are present, for both the orthotropic and the isotropic formulations. Further, coupling is provided by the invariant Γ in the viscosity function $k[\Gamma]$ [see (11)–(13)]. It is therefore clear that TVBO has, in these respects, the same capabilities as the classical plasticity theory.

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APPENDIX: ISOTROPIC VERSION DERIVED FROM THE ORTHOTROPIC VERSION FOR ISOTHERMAL CONDITION

All orthotropic matrices are replaced by isotropic ones. Specifically, (9) and (10) are placed by

$$\mathbf{C}_e = \begin{bmatrix} E & & & & & \\ & E & & & & \\ & & E & & & \\ & & & G & & \\ & & & & G & \\ & & & & & G \end{bmatrix} \tag{A1}$$

and

$$\mathbf{R}_e = \begin{bmatrix} \frac{1-\nu}{\nu_0} & \frac{\nu}{\nu_0} & \frac{\nu}{\nu_0} & & & \\ \frac{\nu}{\nu_0} & \frac{1-\nu}{\nu_0} & \frac{\nu}{\nu_0} & & & \\ \frac{\nu}{\nu_0} & \frac{\nu}{\nu_0} & \frac{1-\nu}{\nu_0} & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \tag{A2}$$

respectively, where $\nu_0 = 1-\nu-2\nu^2$ and $G = E/(2(1+\nu))$. The inelastic modulus matrix $\mathbf{K}_e$ (12) is of the form of (A1) except that K_1 and K_2 replace E and G, respectively. The inelastic Poisson's ratio matrix $\mathbf{R}_k$ is given by (A2) except that η replaces ν. Note that $K_2 = K_1/(2(1+\eta))$ for the isotropic case. In (14), $\mathbf{H}$ and $\mathbf{a}$ are

$$\mathbf{H} = \begin{bmatrix} H_1 & H_3 & H_3 & & & \\ H_3 & H_1 & H_3 & & & \\ H_3 & H_3 & H_1 & & & \\ & & & H_2 & & \\ & & & & H_2 & \\ & & & & & H_2 \end{bmatrix}, \tag{A3}$$

where $H_3 = H_1 - 0.5H_2$ and $\mathbf{a}^t = [a \quad a \quad a \quad 0 \quad 0 \quad 0]$, respectively.

Corresponding expressions hold for $\mathbf{P}$ and $\mathbf{b}$. Also the coefficient of thermal expansion reduces to a scalar.

Next, the theory is reduced to the deviatoric and hydrostatic forms. To this end, the following definitions are introduced:

$$\begin{aligned} \mathbf{e}^{el} &= \boldsymbol{\varepsilon}^{el} - \tfrac{1}{3}\varepsilon_h^{el}\mathbf{I} \\ \mathbf{e}^{in} &= \boldsymbol{\varepsilon}^{in} - \tfrac{1}{3}\varepsilon_h^{in}\mathbf{I} \\ \mathbf{s} &= \boldsymbol{\sigma} - \tfrac{1}{3}\sigma_h\mathbf{I} \\ \mathbf{g}^d &= \mathbf{g} - \tfrac{1}{3}g_h\mathbf{I} \\ \mathbf{f}^d &= \mathbf{f} - \tfrac{1}{3}f_h\mathbf{I} \\ \mathbf{x}^d &= \mathbf{s} - \mathbf{g}^d \end{aligned}$$

with $\mathbf{I}^t = [1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0]$. A variable with subscript h is the hydrostatic component, i.e. $\sigma_h = \sigma_1+\sigma_2+\sigma_3$. With these definitions, (1), (3), (4), (28) and (29) can be written as

$$\dot{\mathbf{e}} = \dot{\mathbf{e}}^{el}+\dot{\mathbf{e}}^{in} \tag{A4}$$

$$\dot{\mathbf{e}}^{el} = \frac{1+\nu}{E}\mathbf{I}_2\dot{\mathbf{s}} \tag{A5}$$

$$\dot{e}_h^{el} = \frac{1-2\nu}{E}\dot{\sigma}_h \tag{A6}$$

$$\dot{\mathbf{e}}^{in} = \frac{1+\eta}{\kappa_1[\Gamma]}\mathbf{I}_2\mathbf{x}^d \tag{A7}$$

$$\dot{\varepsilon}_h^{in} = \frac{1-2\eta}{\kappa_1[\Gamma]}x_h \tag{A8}$$

$$\dot{\mathbf{g}}^d = \frac{\psi[\Gamma]}{E}\dot{\mathbf{s}}+\left(\phi[\Gamma]-\Theta\left(\phi[\Gamma]-E_t\left(1-\frac{\psi[\Gamma]}{E}\right)\right)\right)\frac{\mathbf{x}^d}{\kappa_1[\Gamma]} \tag{A9}$$

$$\dot{g}_h = \frac{\psi[\Gamma]}{E}\dot{\sigma}_h+\left(\phi[\Gamma]-\Theta\left(\phi[\Gamma]-E_t\left(1-\frac{\psi[\Gamma]}{E}\right)\right)\right)\frac{x_h}{\kappa_1[\Gamma]} \tag{A10}$$

$$\dot{\mathbf{f}}^d = E_t\frac{\mathbf{x}^d}{\kappa_1[\Gamma]} \tag{A11}$$

$$\dot{f}_h = E_t\frac{x_h}{\kappa_1[\Gamma]} \tag{A12}$$

where

$$\mathbf{I}_2 = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 2 & & \\ & & & & 2 & \\ & & & & & 2 \end{bmatrix}, \quad \text{and} \quad \kappa_1[\Gamma] = K_1k[\Gamma]$$

and where Γ and Θ are given by (14) and (19), respectively, using the isotropic version of the material matrices and vectors [see (A3)]. When (A3) is used together with $H_2 = 3H_1 = 3$ and $a = 0$, the invariant Γ of (14) reduces to the invariant Γ_0 defined in (7) of Yao and Krempl (1985) or the invariant Γ given in (4) of Krempl and Yao (1987). If

$$P_2 = 3, \quad P_1 = \frac{3}{A^2} \quad \text{and} \quad b = 0,$$

(19) is

$$\Theta = \frac{1}{A}\left(\tfrac{3}{2}(\mathbf{g}^d-\mathbf{f}^d)^t(\mathbf{g}^d-\mathbf{f}^d)\right)^{1/2}. \tag{A13}$$

Substitution of (A13) into (A9) using (A5) and (A7) with $\phi[\Gamma] = \psi[\Gamma]$ yields

$$\dot{\mathbf{g}}^d = \psi[\Gamma]\left(\frac{\dot{\mathbf{e}}^{el}}{1+\nu}+\frac{\dot{\mathbf{e}}^{in}}{1+\eta}\right)-\frac{\psi[\Gamma](1+E_t/E)-E_t}{A}\left(\tfrac{3}{2}(\mathbf{g}^d-\mathbf{f}^d)^t(\mathbf{g}^d-\mathbf{f}^d)\right)^{1/2}\frac{\dot{\mathbf{e}}^{in}}{1+\eta}. \tag{A14}$$

On the other hand the corresponding equation (2) of Krempl and Yao (1987) is

$$\dot{\mathbf{g}}^d = \tfrac{2}{3}\psi[\Gamma]\dot{\mathbf{e}}-\frac{\psi[\Gamma]-E_t^*}{A^*}\left(\tfrac{2}{3}\dot{\mathbf{e}}^{in^t}\dot{\mathbf{e}}^{in}\right)^{1/2}(\mathbf{g}^d-\mathbf{f}^d). \tag{A15}$$

Note that E_t is defined with respect to the inelastic strain and A is defined as the asymptotic value of $g-f$ in uniaxial loading. They are related to E_t^* and A^* by $E_t^* = E_t/B$ and $A^* = A/B$, respectively, with $B = (1+E_t/E)$.

In Yao and Krempl (1985) and Krempl and Yao (1987), the growth law for $\mathbf{f}^d$ is given in total form. The incremental form (31) or (33) was necessary for the thermal case. The present formulation has essentially the same modeling capabilities and is simpler than the total form used previously.